SUGGESTIVE MODEL ANSWERS

Code:- AS-2228

M.A./ M.Sc. (III Sem) Exam. - 2013

MATHEMATICS - (PAPER - II)

(Functional Analysis and its Applications)

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MM - 60

Q # 1(a) :- Let B be a Banach space and N a normed linear space. If $\{T_i\}$ is a non-empty set of continuous linear transformations of B into N with the property that $\{T_i(x)\}$ is a bounded subset of N for each vector x in B, then $\{\|T_i\|\}$ is a bounded set of numbers; i.e., $\{T_i\}$ is bounded as a subset of $\beta(B, N)$.

Q # 1(b) :- $\beta(N, R)$ or $\beta(N, C)$ is ~~defi denoted as a~~ conjugate spaces and denoted by N^*.

If f is a functional on N whose (2)

norm defined by

$$\|f\| = \sup\{\,|f(x)| : \|x\| \le 1\,\}$$

$$= \inf\{\,K : K \ge 0 \text{ and } |f(x)| \le K\|x\| \ \forall x\,\},$$

then N^* is a Banach space.

Q # 1(c) :- l_2 - space's norm

$x = (x_1, x_2, \dots)$ or $x = (x_1, x_2, \dots, x_n)$

$$\|x\| = \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{1/2}$$

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$$

Q # 1(d) :- If $x = (x_1, x_2, \dots, x_n)$ and $y = (y_1, y_2, \dots, y_n)$ be real or complex numbers of n-tuples and its norm is defined by

$$\|x\| = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}, \text{ then}$$

$$\boxed{\sum_{i=1}^{n} |x_i y_i| \le \|x\|.\|y\|.}$$

Q # 1(e) :- l_∞^n - Space.

The space of all n-tuples $x = (x_1, x_2, \cdots, x_n)$ of scalars with norm of x defined by

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_n|\}$$

is called l_∞^n - Space.

Q # 1(f) :- In a H.S. H.

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2 + \langle x, y\rangle + \langle y, x\rangle \text{ for } x, y \in H. \quad (1)$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - \langle x, y\rangle - \langle y, x\rangle . \text{ for } x, y \in H \quad (2)$$

$$\|x+y\|^2 - \|x-y\|^2 = 2\{\langle x, y\rangle + \overline{\langle x, y\rangle}\}$$

$$= 2[2 \operatorname{Re} \langle x, y\rangle]$$

$$= 4 \operatorname{Re} \langle x, y\rangle .$$

Q # 1(g):- Schwartz's inequality

If x and y are any two vectors in a H.S. H, then $|\langle x, y \rangle| \leq \|x\| \cdot \|y\|$.

for proof. assume $z = \frac{y}{\|y\|}$, $y \neq 0$

and using the inequality

$$0 \leq \| x - \langle x, y \rangle y \|^2$$

for required proof.

Q # 1(h):- ~~An~~ An orthonormal set is said to be complete if it is not contained in any larger orthonormal set.

ex. $\{e_1, e_2, e_3\}$ is a completely orthonormal set.

Q # 1 (i):- M is a subset of H.

then $M^{\perp}$ is a closed linear subspace of H.

So that ~~$M^{\perp} =$~~ $M = M^{\perp\perp}$

Also $M^{\perp} = M^{\perp\perp\perp}$

Q # 1(j) :- If M is a closed linear subspace of a H.S. H, then $H = M \oplus M^{\perp}$.

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Q # 3 :- The norm of l_∞-space is

$$\|x\|_\infty = \max\{|x_1|, |x_2|, \ldots, |x_n|, \ldots\}$$

First s.t. $\|x\|_\infty$ is a norm on l_∞-space.

$\therefore$ l_∞-space is a NLS.

For Banach-space construct a Cauchy sequence $\{x_n\}_{n=1}^{\infty}$ of l_∞-space.

Then showing that the sequence $\{x_n\}$ defined on l_∞-space is a convergent sequence. The seq. can be defined by

$$x_m = \left(x_1^m, x_2^m, \ldots, x_n^m\right)$$

$\|x_m - x_p\|_\infty < \epsilon$ for $m, p \geq n_0$

for fixed n_0. Apply completeness of R (or C).

Q # 2(a) :- Two parts needed to be proved.

⇒ Assume M is a complete subspace of a Banach space B. Suppose $\{x_n\}$ be a sequence of M which converges to a point x. Since M is complete so, x is in M. And intern it implies M is closed.

⇐ Suppose M is a closed subspace of a Banach space B.

To prove M is complete. We shall construct a Cauchy sequence $\{x_n\}$ in M. By compling the completeness of B and closedness of M $x_n \to x$ in B.

Let $x \in \overline{M}$. Since M is closed, so $\overline{M} = M$. which implies the cauchy sequence $\{x_n\}$ of M converges to x in M.

Hence M is complete subspace of B.

Q # 2(b) :- Let $\mathcal{H}$ is a Separable H.S.

i.e. it has a countable dense subset D.

Assume K be a orthonormal set in H.

claim:- K is countable.

$\forall\ x, y \in K$ with $x \neq y$,

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2$$
$$= 2.$$

$$\left(\because \begin{array}{l} \|x\| = 1 \\ \|y\| = 1 \end{array} \right)$$

& $x \perp y$

The open ball $B(x, 1/2) = \{z : \|z - x\| < 1/2\}$

$x \in K$.

$\because$ D is dense, so it contain a pt.

in each open ball $B(x, 1/2)$.

If K is uncountable, then D must be

uncountable. ~~and~~

Hence, H is not Separable.

A contradiction.

$\Rightarrow$ K must be countable.

Q # 4(a): (i) $\beta(N)$ is a set of all continuous linear transformation from N into itself.

If $T \in \beta(N)$, then

$$\|T\| = \sup\left\{\frac{\|Tx\|}{\|x\|} : x(\neq 0) \in N\right\}.$$

Defⁿ for $\|T_1 T_2\|$ is

$$\|T_1 T_2\| = \sup\left\{\frac{\|(T_1 T_2)x\|}{\|x\|} , x(\neq 0) \in N\right\}$$

Apply the continuity of $T_1 T_2$ to obtained

$$\boxed{\|T_1 T_2\| \leq \|T_1\| \cdot \|T_2\|}$$

(ii) $T_n \to T$ and $T_n' \to T'$, so

$$\|T_n T_n' - T T'\| = \|T_n T_n' - T_n T' + T_n T' - T T'\|$$

$$= \|T_n (T_n' - T') + (T_n - T) T'\|$$

$$\leq \|T_n (T_n' - T')\| + \|(T_n - T) T'\|$$

$$\leq \|T_n\| \, \|T_n' - T'\| + \|T_n - T\| \, \|T'\|$$

$$\to 0.$$

Hence, $\boxed{T_n T_n' \to T T'}$

Q # 4 (b) :- Let M be a finite dimensional Space. ~~on~~ with dim n.

suppose $\{x_1, x_2, \cdots, x_n\}$ is a basis of M,

then for any $x \in M$,

$$x = \sum_{i=1}^{n} \alpha_i x_i$$

(where α_i : $i = 1, 2, \cdots, n$ are Scalars.)

Let us define two different norms $\|\cdot\|_0$ and $\|\cdot\|$ on M.

We have a theorem that tells us ~~the~~

"If M is finite dimensional on which two norms $\|\cdot\|_1$ & $\|\cdot\|_2$ are defined, then $\exists$ +ive constants m and M s.t.

$$m\|x\|_1 \leq \|x\|_2 \leq M\|x\|_2$$

$\forall x \in M$." and conversely.

So, we need to s.t.

$$m\|x\|_0 \leq \|x\| \leq M\|x\|_0, \forall x \in M.$$

Q # 5 :- We have to show that $(c_0)^* = l_1$. (10)

$$c_0 = \{ x = \{x_n\} : x_n \to 0 \text{ as } n \to \infty \}.$$

Consider a standard basis $\{e_n\}$.

~~Define~~ Let a function f in c^*. which is linear and continuous.

Define a mapping $T : c_0^* \to l_1$ defined by

$$T(f) = (\alpha_1, \alpha_2, \dots, \alpha_n).$$

and finally s.t. $\boxed{\|Tf\| = \|f\|}$.

Q #6 :- Statement

" Let M be a linear subspace of a normed linear space N, and let f be a functional defined on M. Then f can be extended to a functional f_0 defined on the whole space N such that $\|f_0\| = \|f\|$."

We need the following lemma for our main theorem.

Let M be a linear subspace of a NLS N, and let f be a functional defined on M. If x_0 is a vector not in M, and if $M_0 = M + [x_0]$

is a linear subspace spanned by M and x_0, then f can be extended to a functional f_0 defined on M_0 such that $\|f_0\| = \|f\|$.

We need to prove for real (or complex) scalars only.

For the main theorem's proof, first we shall define a partially ordered set and apply Zorn's Lemma. to prove $\|f_0\| = \|f\|$.

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Q # 7(a):- Let H be a Hilbert space, with the inner product $<,>$ defined on H.

We have to obtain the Polarisation identity.

$$4<x,y> = \|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2.$$

$\because$ for any $x,y \in H$

$$\|x+y\|^2 = <x+y, x+y>$$

$$= \|x\|^2 + \|y\|^2 + <x,y> + <y,x>. \quad \text{—— (1)}$$

$$\|x-y\|^2 = \|x\|^2 + \|y\|^2 - <x,y> - <y,x>. \quad \text{—— (2)}$$

From (1) and (2),

$$\|x+y\|^2 - \|x-y\|^2 = 2<x,y> + 2<y,x> \quad \text{—— (3)}$$

By replacing y by iy in (1) and (2), we have

$$\|x+iy\|^2 = \|x\|^2 + \|y\|^2 + <x,iy> + <iy,x> \quad \text{—— (4)}$$

$$\|x-iy\|^2 = \|x\|^2 + \|y\|^2 - <x,iy> - <iy,x> \quad \text{—— (5)}$$

From (4) and (5), (13)

$$\|x+iy\|^2 - \|x-iy\|^2 = 2\langle x, iy\rangle + 2\langle iy, x\rangle$$

$$= -2i\langle x, y\rangle + 2i\langle y, x\rangle$$

$$\Rightarrow i\|x+iy\|^2 - i\|x-iy\|^2$$

$$= 2\langle x, y\rangle - 2\langle y, x\rangle \quad \text{—— (6)}$$

~~Adding~~ (3) and (6), we have the required result.

$$\boxed{\|x+y\|^2 - \|x-y\|^2 + i\|x+iy\|^2 - i\|x-iy\|^2 = 4\langle x, y\rangle.}$$

Q # 7(b) :- Let S be a $\neq \phi$ subset of a H.S. H.

We s.t. $S^{\perp}$ is a closed linear subspace of H.

$$S^{\perp} = \{x \in H : \langle x, y\rangle = 0, \forall y \in S\}$$

First show $S^{\perp}$ is $\neq \phi$.

i.e. ...

construct a seq. $\{x_n\}$ in $S^\perp$ and s.t.

$x_n \to x \in S^\perp$. i.e. $\langle x, y \rangle = 0 \ \forall y \in S$.

Since $S^\perp$ is a closed linear subspace of H. So, it is complete and hence is a H.S.

Q # 8(a):- Let T is a self adjoint operator on a complex H.S. H. i.e. $T^* = T$.

Then $\forall \, x \in H$,

$$\langle Tx, x \rangle = \langle x, T^*x \rangle$$

$$= \langle x, Tx \rangle$$

$$= \overline{\langle Tx, x \rangle} \quad \text{——— (1)}$$

$\Rightarrow \langle Tx, x \rangle$ is real.

For converse, let $\langle Tx, x \rangle$ is real. $\forall x \in H$.

Now, we shall show that T is self adjoint operator. i.e $T^* = T$.

$$\langle Tx, x \rangle = \overline{\langle Tx, x \rangle}$$

$$= \langle x, Tx \rangle$$

$$= \langle T^*x, x \rangle \quad \text{——— (2)}$$

where T^* is a ~~self~~ adjoint operator of T
$\forall\, x \in H$.

From (2),

$$< Tx, x> = < T^*x, x>$$

$$\Rightarrow < Tx - T^*x\, ,\, x> = 0$$

$$\Rightarrow < (T - T^*)x, x> = 0 \quad \forall\, x \in H.$$

$$\Rightarrow \quad T - T^* = 0$$

$$\Rightarrow \quad T = T^*.$$

$\therefore$ T is a self adjoint operator.

Q # 8(b) :- Let P is a projection operator on a closed linear subspace M of H.

Assume $Px = x$.

$\because$ Px is in the range of P and x is in the range of P (by hypothesis).

$\therefore$ $x \in M$.

Conversely, we ~~shall~~ assume that $x \in M$ and will show that $Px = x$. ———— (1)

Suppose $Px = y$. ———— (2)

$$\therefore \quad Px = y \Rightarrow P^2x = Py$$

$$\Rightarrow Px = Py$$

($\because P$ is a proj. operator)

$$\Rightarrow Px - Py = 0$$

$$\Rightarrow P(x-y) = 0$$

$\Rightarrow$ $x - y$ is in the nullspace of P.

$$\Rightarrow x - y \in M^\perp$$

$\therefore$ we can write

$$x - y = z, \quad z \in M^\perp.$$

$$\therefore \quad x = y + z. \qquad \text{———— (3)}$$

$y = Px \Rightarrow y$ is in the range of P. i.e. $y \in M$

Thus the expression $x = y + z$, where $y \in M$ and $z \in M^\perp$.

$\because$ $x \in M$, so we can write

$$x = x + 0 \quad \text{---} \quad (4)$$

$\because$ $\mathcal{H} = M \oplus M^{\perp}$

From (3) and (4), $z = 0 \Rightarrow x = y$.

end.

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